

ON TRIPLE LINES AND CUBIC CURVES — THE ORCHARD PROBLEM REVISITED

GYÖRGY ELEKES AND ENDRE SZABÓ

ABSTRACT. Planar point sets with many triple lines (which contain at least three distinct points of the set) have been studied for 180 years, started with Jackson [8] and followed by Sylvester [11]. Green and Tao [7] has shown recently that the maximum possible number of triple lines for an n element set is $\lfloor n(n-3)/6 \rfloor + 1$. Here we address the related problem of describing the structure of the *asymptotically near-optimal* configurations, i.e., of those for which the number of straight lines, which go through three or more points, has a quadratic (i.e., best possible) order of magnitude. We pose the problem whether such point sets must always be related to cubic curves. To support this conjecture we settle various special cases; some of them (Theorems 2.3 and 4.3) are also related to the four-in-a-line problem of Erdős.

1. INTRODUCTION

Given n point in the plane \mathbb{R}^2 , a line is *3-rich*, if it contains precisely 3 of the given points. One of the oldest problems of combinatorial geometry, the so-called Orchard Problem, is to maximise the number of 3-rich lines (see Jackson [8] and Sylvester [11]). Sylvester showed that the number of 3-rich lines is $n^2/6 + \mathcal{O}(n)$, and recently Green and Tao [7] have found the precise value of the maximum.

Theorem 1.1 (Orchard Problem. Green–Tao). *Suppose that \mathcal{H} is a finite set of n points in the plane. Suppose that $n \geq n_0$ for some sufficiently large absolute constant n_0 . Then there are no more than $\lfloor n(n-3)/6 \rfloor + 1$ lines that are 3-rich, that is they contain precisely 3 points of \mathcal{H} .*

Here we address the related problem of describing the structure of the *asymptotically near-optimal* configurations, i.e., of those for which the number of straight lines, which go through three or more points, has a quadratic (i.e., best possible) order of magnitude.

Definition 1.2. Let \mathcal{H} be a subset of the plane \mathbb{R}^2 . A straight line l is called a *triple line* with respect to \mathcal{H} if there exist three distinct points $P_1, P_2, P_3 \in l \cap \mathcal{H}$. We shall also use the notation

$$\overset{\bullet\bullet\bullet}{\mathcal{H}} \stackrel{\text{def}}{=} \{l ; |l \cap \mathcal{H}| \geq 3\}.$$

We extended the notion of *triple line*, without any change in the definition, to subsets of the projective plane.

Note that $\overset{\bullet\bullet\bullet}{\mathcal{H}}$ is a set of *lines*, not a set of *triples*; e.g. if \mathcal{H} is a collinear set of 3 or more points then $|\overset{\bullet\bullet\bullet}{\mathcal{H}}| = 1$.

Elekes is partially supported by HU-NSF grants OTKA T014105 T014302 and T019367.
Szabó is partially supported by OTKA grants NK81203 and K84233.

Triple lines are not necessarily 3-rich (as they may be 4-rich, 5-rich, and so on), hence Theorem 1.1 does not directly bound the size of \mathcal{H} . In any case, it is easy to find a (non-sharp) quadric upper bound. Indeed, each line with three points contains three segments of the $\binom{n}{2}$ which connect pairs of points of \mathcal{H} , hence

$$|\mathcal{H}| \leq \frac{1}{3} \binom{n}{2} = \frac{n^2}{6} - n/6.$$

The following examples show four simple configurations for which the quadratic order of magnitude can really be attained. Two of them consist of three collinear point sets each, the third one is located on a conic and a straight line, while the fourth one on a cubic.

Example 1.3. If $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3$ are three copies of an arithmetic progression on three equidistant parallel lines then $|\overline{\mathcal{H}_1 \mathcal{H}_2 \mathcal{H}_3}| \approx N^2/18$, where N denotes the total number of points and $\overline{\mathcal{H}_1 \mathcal{H}_2 \mathcal{H}_3}$ denotes the set of lines l such that there exist three distinct points $P_i \in l \cap \mathcal{H}_i$ for $i = 1, 2, 3$.

(It is slightly better to place a point set of “double density” on the middle line.)

Example 1.4. Let P_1, P_2, P_3 be the vertices of a non-degenerate triangle, and \mathcal{H}_i ($i = 1, 2, 3$) point sets on the line through the vertices P_{i-1} and P_{i+1} , defined by

$$(1) \quad \mathcal{H}_i = \left\{ X ; \frac{\overline{P_{i-1}X}}{\overline{XP_{i+1}}} \in \{\pm 1, \pm 2^{\pm 1}, \pm 4^{\pm 1}, \dots, \pm 2^{\pm(n-1)}\} \right\},$$

where $i \pm 1$ is used mod 3 in the indices of the P_i . (See Figure 1.)

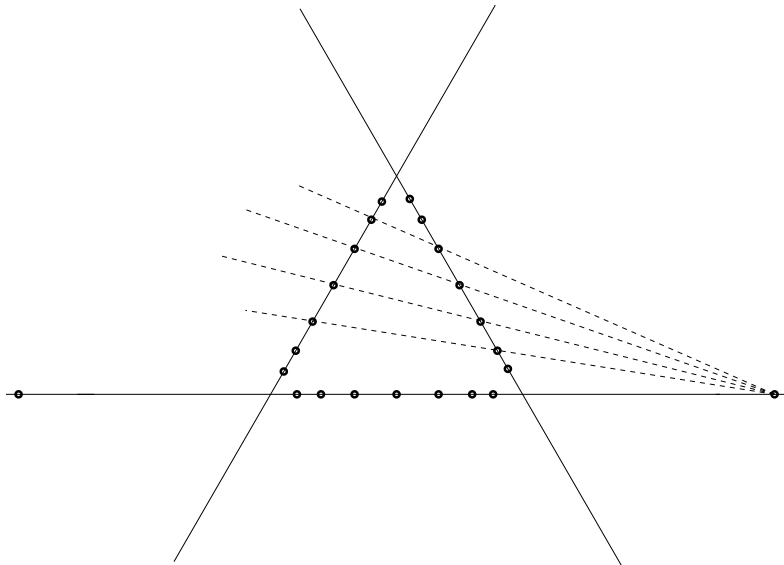


FIGURE 1. Portion of a triangular configuration with some triple lines marked.

Here again $|\overline{\mathcal{H}_1 \mathcal{H}_2 \mathcal{H}_3}| \approx N^2/18$, where N denotes the total number of points. (The observant reader may have noticed that we allowed (-1) among the ratios, i.e., X may be a point at infinity.)

Example 1.5. The $\binom{n}{2}$ segments which connect pairs of vertices of a regular n -gon C only determine n distinct slopes. Let D be the set of points on the line at infinity which correspond to these directions. Then $|\overset{\bullet\bullet\bullet}{\overline{CD}}| \approx N^2/8$, where $N = |C \cup D| = 2n$ and $\overset{\bullet\bullet\bullet}{\overline{CD}}$ stands for $\overset{\bullet\bullet\bullet}{\overline{CCD}}$.

Example 1.6. The point set $\mathcal{H} = \{(i, i^3) ; i = -n, \dots, n\}$ on the curve $y = x^3$ satisfies $|\overset{\bullet\bullet\bullet}{\mathcal{H}}| \approx N^2/8$, where $N = 2n + 1$. This can easily be demonstrated by making use of the fact that three points (a, a^3) , (b, b^3) and (c, c^3) are collinear iff $a + b + c = 0$.

The goal of this paper is to show that point sets with many triple lines are, from several points of view, closely related to cubics.

2. PROBLEMS AND RESULTS

A conjecture. Since all the above examples with a quadratic order of magnitude of the triple lines involve cubic curves (some of which are degenerate), it is natural to believe the following.

Conjecture 2.1. *If $|\overset{\bullet\bullet\bullet}{\mathcal{H}}| \geq c|\mathcal{H}|^2$ then ten or more points of \mathcal{H} lie on a (possibly degenerate) cubic, provided that $|\mathcal{H}| > n_0(c)$.*

Here the “magic number” 10 is the least non-trivial value since any nine points of \mathbb{R}^2 lie on a cubic. Perhaps even a stronger version may hold: *for every $c > 0$ and positive integer k there exist $c^* = c^*(c, k) > 0$ and $n_0 = n_0(c, k)$, such that, if $|\overset{\bullet\bullet\bullet}{\mathcal{H}}| \geq c|\mathcal{H}|^2$ then there is a con-cubic $\mathcal{H}^* \subset \mathcal{H}$ with $|\mathcal{H}^*| \geq k$ and $|\overset{\bullet\bullet\bullet}{\mathcal{H}^*}| \geq c^*|\mathcal{H}^*|^2$, provided that $|\mathcal{H}| \geq n_0$.*

It is very likely that in place of k above, even $c^*|\mathcal{H}|^\alpha$ con-cubic points exist (for some $c^* = c^*(c) > 0$ and $\alpha = \alpha(c) > 0$). An example with only $O(\sqrt{|\mathcal{H}|})$ such points is a $k \times k$ square or parallelogram lattice where the points of three parallel lines provide the set located on a (degenerate) cubic. Similarly, projections of d dimensional cube lattices to \mathbb{R}^2 form structures with only $O(|\mathcal{H}|^{1/d})$ con-cubic points.

Moreover, if we assume that \mathcal{H} has no four-in-a-line and $|\overset{\bullet\bullet\bullet}{\mathcal{H}}| \geq c|\mathcal{H}|^2$, then perhaps as many as $c^*|\mathcal{H}|$ of its points will lie on an irreducible cubic.

Results. In order to support the above conjecture, we settle various special cases in the affirmative. Our main result is the following.

Theorem 2.2. *In \mathbb{R}^2 , if irreducible algebraic curve of degree d contains a set \mathcal{H} of n points with $|\overset{\bullet\bullet\bullet}{\mathcal{H}}| \geq cn^2$ then the curve is a cubic — provided that $n > n_0(c, d)$.*

Two simple applications of the forthcoming slightly more general Theorem 4.1 are the following.

Theorem 2.3. *In \mathbb{R}^2 , no irreducible algebraic curve of degree d can accommodate n points with cn^2 quadruple lines if $n > n_0(c, d)$.*

Theorem 2.4. *In \mathbb{R}^2 , if a set of n points located on an irreducible algebraic curve of degree d only determines Cn distinct directions then the curve is a conic — provided that $n > n_0(d, C)$.*

The above theorems are of algebraic geometric nature, therefore it is natural to ask analogous questions in complex geometry (i.e. when the point set and the algebraic curves live in \mathbb{C}^2). However, in this paper we restrict our attention to the real plane \mathbb{R}^2 .

In some other results (see Section 5) we allow part of the points (a positive proportion) to be arbitrary and only restrict the rest of them to a conic. In this case it will turn out that a large subset of the first part must be collinear. (Here again, the conic and the straight line, together, form a degenerate cubic.) The following is the essence of Theorems 5.1 and 5.2.

*Let $\mathcal{H} = \mathcal{H}_1 \cup \mathcal{H}_2$ and assume that \mathcal{H}_1 lies on a (possibly degenerate) conic Γ while $\mathcal{H}_2 \cap \Gamma = \emptyset$. If $n \leq |\mathcal{H}_1|, |\mathcal{H}_2| \leq Cn$ and $|\overline{\mathcal{H}_1 \mathcal{H}_2}| \geq cn^2$ then some c^*n points of \mathcal{H}_2 are collinear. (Here $c^* = c^*(c, C)$ does not depend on n .)*

We also mention a theorem of Jamison [9] which can be considered as another result in the direction of our Conjecture 2.1: if the diagonals and sides of a convex n -gon only determine n distinct slopes (which is smallest possible), then the vertices of the polygon all lie on an ellipse. In terms of triple lines (and a degenerate cubic formed by a straight line and an ellipse) this can be formulated as follows:

(Jamison's Theorem) if \mathcal{H}_1 is the vertex set of a convex polygon and \mathcal{H}_2 lies on the line at infinity with $|\mathcal{H}_1| = |\mathcal{H}_2| = n$ then $|\overline{\mathcal{H}_1 \mathcal{H}_2}| = \binom{n}{2}$ implies that \mathcal{H}_1 lies on an ellipse.

A similar statement was proven by Wettl [12] for finite projective planes.

The structure of the paper. The aforementioned results (usually in stronger form) are presented in detail in the last two sections. Before that, we list some basic facts on the relation between continuous curves, collinearity and Abelian groups, concluding in the fundamental observation Lemma 3.8.

3. COLLINEARITY AND GROUPS

Collinearity on cubics.

Definition 3.1. Let $\Gamma_1, \Gamma_2, \Gamma_3$ be three (not necessarily distinct) Jordan curves (i.e., bijective continuous images of an interval or a circle) in the projective plane, and $\langle \mathcal{A}, \oplus \rangle$ an Abelian topological group. We say that *collinearity between Γ_1, Γ_2 and Γ_3 can be described by the group operation \oplus* , if, for $i = 1, 2, 3$, there are homeomorphic monomorphisms (i.e., continuous injections whose inverses are also continuous)

$$f_i : \Gamma_i \rightarrow \mathcal{A}$$

— in other words, “parametrisation” of the Γ_i with \mathcal{A} — such that three distinct points $P_1 \in \Gamma_1, P_2 \in \Gamma_2, P_3 \in \Gamma_3$ are collinear if and only if

$$f_1(P_1) \oplus f_2(P_2) \oplus f_3(P_3) = 0 \in \mathcal{A}.$$

The curves we consider will usually be irreducible components of algebraic curves in \mathbb{R}^2 — or subsets thereof. However, sometimes we must also study general continuous curves, as well.

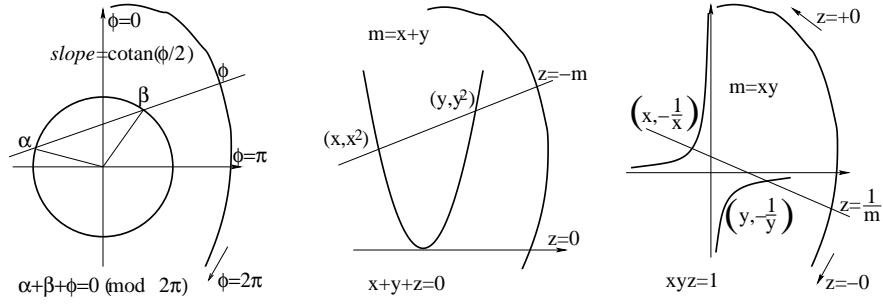


FIGURE 2. Parametrisation of reducible cubics: a conic plus the line at infinity. (Due to lack of sufficient space the line at infinity is depicted as a bent curve.)

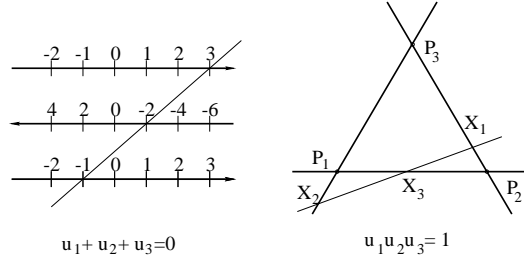


FIGURE 3. Parametrisation of reducible cubics: three straight lines. In case of a triangle, $u_i \stackrel{\text{def}}{=} f(X_i) = \overline{X_i P_{i-1}} / \overline{X_i P_{i+1}}$.

In what follows we denote the set of regular points of an algebraic curve Γ by $\text{Reg}(\Gamma)$. The connected components of $\text{Reg}(\Gamma)$ are Jordan curves.

Proposition 3.2. *Let \mathcal{C} be a cubic curve in the projective plane. If $\Gamma_1, \Gamma_2, \Gamma_3$ are (not necessarily distinct) connected components of $\text{Reg}(\mathcal{C})$, then collinearity between them can be described by commutative group operation — unless two of the Γ_i are identical straight lines.*

Indeed, for reducible cubics, Figures 2 and 3 show appropriate parametrisation in the real plane. (Any other reducible cubic is projective equivalent to one of these.) The groups used are $\langle \mathbb{R}, + \rangle / 2\pi\mathbb{Z}$, $\langle \mathbb{R}, + \rangle$, $\langle \mathbb{R} \setminus \{0\}, \cdot \rangle$ in Figure 2 and $\langle \mathbb{R}, + \rangle$, $\langle \mathbb{R} \setminus \{0\}, \cdot \rangle$ in Figure 3, respectively. If $\Gamma_1 = \Gamma_2 = \Gamma_3 = \mathcal{C} = \{(x, x^3) ; x \in \mathbb{R}\}$ then the parametrisation $f(x, x^3) = x$ works well. It is also well-known that for irreducible cubics (i.e. elliptic curves), suitable parametrisation exist (see, e.g., in [10]).

Remark 3.3. Note that in all cases only regular points are parametrised. This will make no confusion since singular (e.g., multiple) points of a cubic never occur in proper collinear triples.

Collinearity on continuous curves. Throughout this section we consider the graphs of three continuous real functions.

Definition 3.4. We call α, β and γ a *standard system of continuous real functions* if

- (i) they are defined in a neighbourhood \mathcal{D} of 0;
- (ii) $\alpha(x) < \beta(x) < \gamma(x)$ for all $x \in \mathcal{D}$;
- (iii) any straight line through any point of the graph of any of the three functions intersects the other two graphs in at most one point each.

For such functions α , β and γ we denote their graphs (which are Jordan arcs) by $\overline{\alpha}$, $\overline{\beta}$ and $\overline{\gamma}$.

Remark 3.5. Assumption (iii) is not very strong a requirement; e.g., if the functions are differentiable at 0 (elsewhere they may not even be smooth) then \mathcal{D} can be restricted to a sufficiently small neighbourhood of 0 so that (iii) be satisfied there.

Proposition 3.6. *Let $P(x, \beta(x))$ be a point of the “middle” graph $\overline{\beta}$. Connect it with lines to the two points $A_0(0, \alpha(0))$ and $C_0(0, \gamma(0))$; moreover, denote by $C(P)$ and $A(P)$ the points of intersection of these lines with the graphs $\overline{\gamma}$ and $\overline{\alpha}$, respectively (if they exist). Finally, let $B(P)$ be the intersection of the line through $A(P)$ and $C(P)$ with the graph $\overline{\beta}$. Then*

- (i) *if x is sufficiently close to 0 then $A(P)$, $B(P)$ and $C(P)$ really exist; and the composite mappings*

$$x \mapsto P = P(x, \beta(x)) \mapsto \begin{cases} A(P) \text{ or} \\ B(P) \text{ or} \\ C(P) \end{cases}$$

are continuous functions $\mathbb{R} \rightarrow \mathbb{R}^2$;

- (ii) *for every point \hat{B} of the graph $\overline{\beta}$, sufficiently close to the y -axis, there is a P for which $\hat{B} = B(P)$.*

The straightforward proof using straightforward calculus — together with the Intermediate Value Theorem for (ii) — is left to the reader. ■

Next we shall study when will collinearity between $\overline{\alpha}$, $\overline{\beta}$ and $\overline{\gamma}$ be described by an Abelian topological group \mathcal{A} , so we will search for parametrisations $f_\alpha : \overline{\alpha} \rightarrow \mathcal{A}$, $f_\beta : \overline{\beta} \rightarrow \mathcal{A}$ and $f_\gamma : \overline{\gamma} \rightarrow \mathcal{A}$. Part (iii) of Definition 3.4 also implies that the curves $\overline{\alpha}$, $\overline{\beta}$ and $\overline{\gamma}$ must be pairwise disjoint. That is why, in what follows, we shall only use one notation

$$f := (f_\alpha \cup f_\beta \cup f_\gamma) : (\overline{\alpha} \cup \overline{\beta} \cup \overline{\gamma}) \rightarrow \mathcal{A}$$

in place of three.

Lemma 3.7 (“Parameter-halving lemma”). *Let α , β and γ form a standard system of continuous real functions. Moreover, let $B_0 = (0, \beta(0))$ and $A_0, C_0, P, A = A(P), B = B(P)$ and $C = C(P)$ be as above. Assume that collinearity between the three graphs is described by a group operation $\langle \mathcal{A}, \oplus \rangle$ and mapping (parametrisation) f . Then*

- (i) *if*

$$\begin{aligned} f(P) &= f(B_0) \oplus p \quad \text{and} \\ f(B) &= f(B_0) \oplus b \end{aligned}$$

then $p = b/2$, i.e., $b = p \oplus p$.

- (ii) *if B is sufficiently close to B_0 then there really exists a P for which $f(P) = f(B_0) \oplus b/2$.*

Proof (i) Note that

$$f(A_0) \oplus f(B_0) \oplus f(C_0) = 0 \in \mathcal{A}.$$

Moreover, the collinearity of the triples C_0PA and CPA_0 imply

$$\begin{aligned} f(A) &= f(A_0) \ominus p; \\ f(C) &= f(C_0) \ominus p, \end{aligned}$$

respectively; therefore

$$\begin{aligned} f(B) &= p \oplus p \ominus f(A_0) \ominus f(C_0) = \\ &= p \oplus p \oplus f(B_0), \end{aligned}$$

whence the required identity.

(ii) is obvious from Proposition 3.6(ii). ■

A fundamental lemma. The forthcoming Lemma 3.8 will work as our first tool for proving Theorem 2.2 and the slightly more general Theorem 4.1. The basic idea is to use the well-known construction of the group structure on cubics. If we know a few points on a cubic, then just by drawing specific lines and marking specific intersection points we can construct infinitely many new points on that cubic.

The essence of the following statement is that only on cubics can Abelian groups describe collinearity.

Lemma 3.8. *Let α, β, γ be a standard system of continuous functions defined in a neighbourhood of 0. Assume that collinearity between the three graphs is described by a group operation. Then their union $\overline{\alpha} \cup \overline{\beta} \cup \overline{\gamma}$ is contained in a (possibly reducible) cubic.*

For the proof we need certain special structures; they will be the topic of the next subsection. The proof itself comes then in the subsection afterwards.

Ten point configurations and cantilevers. Two types of point-line configurations will play special roles in what follows. The first one consists of ten points and a certain structure of triple lines while the latter will extend the former one.

Given $\overline{\alpha}, \overline{\beta}, \overline{\gamma}$ as in Lemma 3.8, we define ten point configurations as follows.

Denote, again, by A_0, B_0 and C_0 the points of intersection of the y -axis with the three graphs, respectively.

Choose B_1 on $\overline{\beta}$ sufficiently close to B_0 in order to make sure that all the forthcoming points exist. (This will be described later in more detail.) Let A_1 (resp. C_1) be the point of intersection of $\overline{\alpha}$ with the line through B_1 and C_0 (resp. that of $\overline{\gamma}$ with the line through B_1 and A_0). Define B_2 to be the point of intersection of $\overline{\beta}$ with the line through A_1 and C_1 . Let A_2 (resp. C_2) be the point of intersection of $\overline{\alpha}$ with the line through B_2 and C_0 (resp. that of $\overline{\gamma}$ with the line through B_2 and A_0).

The definition of B_3 is asymmetric: it will be the intersection of $\overline{\beta}$ with the line through A_1 and C_2 . Finally, B_4 is, again, defined in a symmetric manner: the intersection of $\overline{\beta}$ with the line through A_2 and C_2 (see Figure 4). Note that by iterated application of Proposition 3.6, the rest of the points will all exist if B_1 is close enough to B_0 .

The observant reader may have noticed that we defined eleven points altogether (instead of just ten). However, B_0 will NOT be in our configuration.

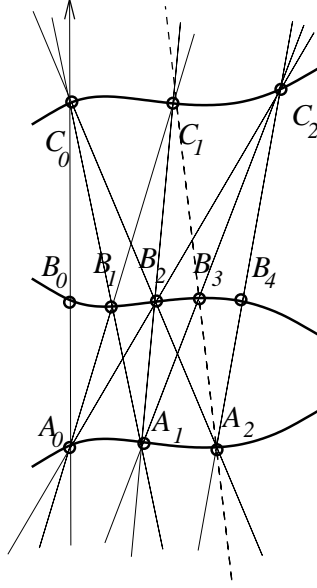


FIGURE 4. The straight line $A_2B_3C_1$ is not used in the definition of the points.

Definition 3.9. Given $\bar{\alpha}, \bar{\beta}, \bar{\gamma}$ as in Lemma 3.8, we call the above

$$\langle A_0, A_1, A_2, B_1, B_2, B_3, B_4, C_0, C_1, C_2 \rangle$$

a *ten point configuration* defined by B_1 .

Proposition 3.10. If α, β, γ is a standard system of continuous real functions and collinearity between their graphs is described by $\langle \mathcal{A}, \oplus \rangle$ and mapping f then

- (i) A_2, B_3 and C_1 are collinear.
- (ii) More generally, A_i, B_j and C_k are collinear iff $i + k = j$.
- (iii) There is a $\Delta \in \mathcal{A}$ such that $f(A_i) = f(A_0) \oplus i\Delta$, $f(B_i) = f(B_0) \oplus i\Delta$, and $f(C_i) = f(C_0) \oplus i\Delta$.

Proof Indeed, statement (ii) — with the exception of (i) — holds by definition. For $\Delta \stackrel{\text{def}}{=} f(B_1) \oplus f(B_0)$, this implies statement (iii) by group identities. Finally, (i) follows from (iii), using $f(A_0) \oplus f(B_0) \oplus f(C_0) = 0$, which, together with (iii), implies $f(A_2) \oplus f(B_3) \oplus f(C_1) = 0$. ■

Lemma 3.11 (Ten point Lemma). Let $\bar{\alpha}, \bar{\beta}, \bar{\gamma}$ be as in Lemma 3.8. Assume, moreover, that a ten point configuration defined on them is contained in two (possibly reducible) cubics \mathcal{C}_1 and \mathcal{C}_2 . Then $\mathcal{C}_1 = \mathcal{C}_2$.

Proof According to the definition of a standard system of continuous functions, if a straight line l contains two points of any of the three graphs then l is disjoint from the other two. This leaves us three possibilities for a cubic \mathcal{C}_j ($j = 1, 2$):

- Type 1. three straight lines, one through the A_i , one through the B_i , and one through the C_i ;
- Type 2. a straight line through all (three or four) points of one of the graphs and a non-degenerate conic through the rest of them;

Type 3. an irreducible cubic through all the points.

According to Bézout's Theorem [6], two distinct irreducible algebraic curves of degree k and m , respectively, can only intersect in at most km points. This immediately implies the Lemma. Indeed, if we assume $\mathcal{C}_1 \neq \mathcal{C}_2$ for a contradiction, then e.g., if \mathcal{C}_1 is of type 2 and \mathcal{C}_2 of type 3 then either \mathcal{C}_2 and a straight line component of \mathcal{C}_1 intersect in four or more points, or \mathcal{C}_2 and a conic component of \mathcal{C}_1 intersect in seven or more points — a contradiction anyway. (The other pairs of types are easier.) ■

Lemma 3.12 (Nine Point Lemma). *Let $\overline{\alpha}, \overline{\beta}, \overline{\gamma}$ be as in Lemma 3.8, consider a ten point configuration on them. If a (possibly reducible) cubic \mathcal{C} contains, with the exception of B_3 , the other nine points, then it must also contain B_3 . Moreover, all ten points must belong to $\text{Reg}(\mathcal{C})$.*

Proof Define $\delta \stackrel{\text{def}}{=} f(A_0) \oplus f(B_1) \oplus f(C_0) \in \mathcal{A}$. Then $\delta \neq 0$ since A_0, B_1 and C_0 are not collinear. What is $X \in \overline{\beta}$ for which $f(X) = 3\delta$? According to Proposition 3.10, it must be the point of intersection of the two straight lines $\overline{C_1A_2}$ and $\overline{C_2A_1}$. Finally, lines passing through a singular point $P \in \mathcal{C}$, if it has any, may contain at most two points of \mathcal{C} , so the lines in our ten point configuration may not pass through P . In particular, P cannot belong to a ten point configuration. ■

Remark 3.13. Note that Lemmas 3.11 and 3.12 also imply that two cubics must coincide if they both contain the nine points (with the exception of B_3). However, we shall not need this fact.

Now we extend ten point configurations to what we call “cantilevers”.¹ (We hope that the shape of these structures will really justify this non-conventional notion.)

Starting from a ten point configuration on α, β, γ , we proceed recursively as follows.

Assume that B_i and B_{i+1} have already been defined for an $i \geq 3$. Then let C_i be the intersection of the lines $\overline{A_0B_i}$ and $\overline{A_1B_{i+1}}$ while A_i the intersection of the lines $\overline{C_0B_i}$ and $\overline{C_1B_{i+1}}$. Finally, define B_{i+2} to be the intersection of $\overline{A_2C_i}$ and $\overline{C_2A_i}$. (See Figure 5.) It is important to note that the construction of cantilevers use only the ten points, and does not depend on the three curves.

Remark 3.14. Formally, here we work in the projective plane and even allow points of intersection located on the line at infinity. However, whenever we apply this construction, all points will lie on the curves $\overline{\alpha}, \overline{\beta}$, and $\overline{\gamma}$.

Lemma 3.15. *If the straight lines $\overline{A_0B_i}$ and $\overline{A_1B_{i+1}}$ intersect $\overline{\gamma}$ then this must happen at C_i , and similarly for $\overline{C_0B_i}, \overline{C_1B_{i+1}}, \overline{\alpha}$ and A_i . Moreover, if the above intersections all exist (and coincide with the C_i and the A_i , respectively), then B_{i+2} is located on $\overline{\beta}$.*

Proof Denote by X and Y the points of intersection of $\overline{\gamma}$ with $\overline{A_0B_i}$ and $\overline{A_1B_{i+1}}$, respectively. What is $f(X)$ then? By Proposition 3.10,

$$f(X) = \ominus f(A_0) \ominus f(B_i) = \ominus f(A_0) \ominus f(B_0) \ominus i\Delta = f(C_0) \ominus i\Delta.$$

¹Cantilever [noun]: a projecting beam or structure supported only at one end. (The Merriam–Webster Dictionary).

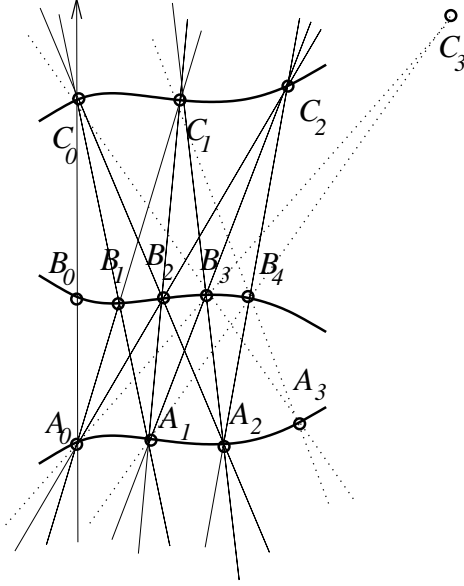


FIGURE 5.

Similarly, $f(Y) = f(C_0) \ominus (i+1-1)\Delta = f(X)$, whence $X = Y$. Therefore, also C_i must coincide with these points.

A similar argument proves the statement on B_{i+2} , too, since in that case the lines which define it must always intersect $\bar{\beta}$. ■

Lemma 3.16. *If a cubic \mathcal{C} contains the nine points $A_0, A_1, A_2, B_1, B_2, B_4, C_0, C_1, C_2$ of a ten point configuration then the entire cantilever (of infinite length) built from this configuration is contained in $\text{Reg}(\mathcal{C})$.*

Proof By Lemma 3.12 the entire ten point configuration is contained in $\text{Reg}(\mathcal{C})$. Let Γ_1, Γ_2 , and Γ_3 denote the connected components of $\text{Reg}(\mathcal{C})$ containing A_0, B_1 , and C_0 , respectively. By Proposition 3.2 the collinearity between the Γ_i is described by a group operation, let f_1, f_2, f_3 denote the parametrisations. In this case (i.e. for cubics) all f_i are bijections, hence they have inverse functions.

Consider the group element $\Delta = f_3(C_1) \ominus f_3(C_0)$. For all $n \geq 0$ we define the following points on \mathcal{C} :

$$\begin{aligned} A'_n &= f_1^{-1}(f_1(A_0) \ominus n\Delta) \\ B'_n &= f_2^{-1}(f_2(B_1) \oplus (n-1)\Delta) \\ C'_n &= f_3^{-1}(f_3(C_1) \ominus (n-1)\Delta) \end{aligned}$$

Plugging in $n = 0$ and $n = 1$ we obtain that

$$A'_0 = A_0, \quad B'_1 = B_1, \quad C'_0 = C_0, \quad C'_1 = C_1.$$

By assumption A_0, B_1, C_2 are collinear, hence $f_1(A_0) \oplus f_2(B_1) \oplus f_3(C_2) = 0$. This implies that

$$f_1(A'_i) \oplus f_2(B'_j) \oplus f_3(C'_k) = \ominus i\Delta \oplus (j-1)\Delta \oplus (k-1)\Delta = (i+k-j)\Delta$$

hence A'_i, B'_j, C'_k are collinear iff $i+k=j$.

Moreover, if a line can intersect \mathcal{C} in at most three points, and if two of the intersection points are regular then all of them must be regular. Apply this to the line $\overline{C_0 B_1} = \overline{C'_0 B'_1}$. The third intersection point of this line with $\text{Reg}(\mathcal{C})$ must be A_1 by Proposition 3.10, but above we proved it is A'_1 . Therefore $A'_1 = A_1$. Similarly, the third intersection point of the line $\overline{A_1 C_1} = \overline{A'_1 C'_1}$ with $\text{Reg}(\mathcal{C})$ must be B_2 on the one hand, and B'_2 on the other hand, which implies $B_2 = B'_2$. Finally apply the same argument to the lines $\overline{C_0 B_2} = \overline{C'_0 B'_2}$ and $\overline{A_0 B_2} = \overline{A'_0 B'_2}$ to obtain that $A_2 = A'_2$ and $C_2 = C'_2$.

To prove the lemma it is enough to show that $A'_n = A_n$, $B'_n = B_n$ and $C'_n = C_n$ for all $n \geq 1$. We prove it by induction on n . However, it is easier to do the induction with a slightly stronger statement. So we shall prove that

$$A'_n = A_n, \quad B'_{n+1} = B_{n+1}, \quad B'_{n+2} = B_{n+2}, \quad C'_n = C_n$$

for all $n \geq 0$. For $n = 0$ we have already seen this. Assume now that it is true for $n - 1$. Consider the intersection point of the lines $\overline{C_0 B_{n+1}} = \overline{C'_0 B'_{n+1}}$ and $\overline{C_1 B_{n+2}} = \overline{C'_1 B'_{n+2}}$. On the one hand it must be A_{n+1} , on the other hand it is A'_{n+1} , hence $A'_{n+1} = A_{n+1}$. Similarly, the intersection point of the lines $\overline{A_0 B_{n+1}} = \overline{A'_0 B'_{n+1}}$ and $\overline{A_1 B_{n+2}} = \overline{A'_1 B'_{n+2}}$ must be $C'_{n+1} = C_{n+1}$. Finally, the intersection point of $\overline{C_2 A_{n+1}} = \overline{C'_2 A'_{n+1}}$ and $\overline{A_2 C_{n+1}} = \overline{A'_2 C'_{n+1}}$ must be $B_{n+3} = B'_{n+3}$. This completes the induction step. ■

Proof of Lemma 3.8. It suffices to show that, for any x_0 in the (common) domain \mathcal{D} of the functions α , β , and γ , there exists a cubic \mathcal{C} which contains the three graphs *restricted to a sufficiently small neighbourhood* of x_0 . Indeed, if we have such a neighbourhood (for each x_0) then it is possible to extend any of them as follows. Let $x_1 \in \mathcal{D}$ be one of the endpoints of this neighbourhood (interval) and consider a cubic \mathcal{C}_1 which contains the graphs in a neighbourhood of x_1 . Within the intersection of the two intervals one can find a ten point configuration contained both by \mathcal{C} and \mathcal{C}_1 . By the Ten Point Lemma (Lemma 3.11), $\mathcal{C} = \mathcal{C}_1$, i.e., we have a longer neighbourhood of x_0 . Thus the *maximal* such neighbourhood must be \mathcal{D} itself.

Now we find an appropriate cubic in a neighbourhood of (without loss of generality) $x_0 = 0$. To start with, we select a ten point configuration, also include B_0 , and extend it to the other side as follows. Start “backwards” from the collinear triple A_2, B_4, C_2 and define (using B_3 in place of the original B_1) a $5 + 9 + 5$ point cantilever — with A_0, B_0 and C_0 in the “middle”. We shall denote this structure by \mathcal{H} .

Define $B_{1/2}$ as in the Parameter Halving Lemma (Lemma 3.7) and, starting from A_0, B_0 and C_0 , using this $B_{1/2}$ as reference point, define a cantilever with points A_i ($i = 0, \dots, 4$), B_i ($i = 0, \dots, 8$) and C_i ($i = 0, \dots, 4$). Of course, the new points will include the old ones, as well, by Proposition 3.10(iii). Also continue the structure “to the left” and denote this refined (halved) cantilever of 35 points by \mathcal{H}_1 . Keep on defining $B_{1/2^n}$ and \mathcal{H}_n by recursive halving, where the latter consists of $(2^{n+2} + 1) + (2^{n+3} + 1) + (2^{n+4} + 1) = 2^{n+4} + 3$ points.

For each n , consider a cubic \mathcal{C}_n which passes through $A_0, A_{1/2^n}, A_{2/2^n}, B_{1/2^n}, B_{2/2^n}, B_{4/2^n}, C_0, C_{1/2^n}$, and $C_{2/2^n}$. By Lemma 3.16 this cubic contains all points of \mathcal{H}_n . In particular, all \mathcal{C}_n must contain the ten point configuration we started with, hence all these cubics are identical by the Ten Point Lemma (Lemma 3.11).

At this point we have a cubic \mathcal{C} for which

$$\bigcup_n \mathcal{H}_n \subset \mathcal{C}.$$

On it, the halving process (starting from \mathcal{H}_0) gives exactly the same \mathcal{H}_n , whence the parameters which occur in $\bigcup_n \mathcal{H}_n$ are dense somewhere in an open set \mathcal{U} of the topological group A . Hence so is the point set itself in three corresponding arcs of \mathcal{C} (i.e., in the homeomorphic pre-images of \mathcal{U}). By the continuity of α, β, γ (and $\bigcup_n \mathcal{H}_n \subset \mathcal{C}$), these arcs are completely on \mathcal{C} , as well, thus providing the required common parts. ■

Surfaces and groups. Let $F \in \mathbb{R}[x, y, z]$ be a polynomial of three real variables. Denote by

$$S = S_F \stackrel{\text{def}}{=} \{(x, y, z) \in \mathbb{R}^3 ; F(x, y, z) = 0\}$$

its zero set, i.e., the algebraic surface described by the equation $F = 0$. The degree of S_F is the (total) degree of its defining polynomial F .

Definition 3.17. We say that a surface $S \subset \mathbb{R}^3$ is *described by a commutative group operation* $\langle \mathcal{A}, \oplus \rangle$ if there are mappings (“parametrisations”) $f_i : \mathbb{R} \mapsto \mathcal{A}$ for $i = 1, 2, 3$ such that

$$(x_1, x_2, x_3) \in S \Leftrightarrow f_1(x_1) \oplus f_2(x_2) \oplus f_3(x_3) = 0.$$

E.g., the ball of equation $x^2 + y^2 + z^2 = 1$ is described by the additive group through the mappings $f_i(t) = t^2 - 1/3$ ($i = 1, 2, 3$).

One of the main ingredients of our proof is Theorem 3.18 below, proven in [4].

Assume we consider a plane $\alpha x + \beta y + \gamma z = \delta$, intersecting the cube $[0, n]^3$. If the coefficients $\alpha, \beta, \gamma, \delta$ are rationals with small numerators and denominators then this plane will contain $\sim n^2$ lattice points. If we apply independent univariate transformations in the three coordinates, x, y, z , then we can easily produce 2-dimensional surfaces — described by some equation $f(x) + g(y) + h(z) = \delta$ — containing a quadratic number of points from a product set $X \times Y \times Z$, where $|X| = |Y| = |Z| = n$. The main result of [4] asserts that if some appropriate algebraicity conditions hold then (apart from being a cylinder) this is the only way for a surface $F(x, y, z) = 0$ to contain a near-quadratic number of points from such a product set $X \times Y \times Z$.

As usual, we call a function of one or two variable(s) *analytic* at a point if it can be expressed as a convergent power series in a neighbourhood. Also, it is analytic on an open set if it is analytic at each of its points.

Theorem 3.18 (“Surface Theorem”, see [4], Theorem 3.). *For any positive integer d there exist positive constants $\eta = \eta(c, d)$, $\lambda = \lambda(c, d)$ and $n_0 = n_0(c, d)$ with the following property.*

If $V \subset \mathbb{R}^3$ is an algebraic surface (i.e. each component is two dimensional) of degree $\leq d$ then the following are equivalent:

- (a) *For at least one $n > n_0(c, d)$ there exist $X, Y, Z \subset \mathbb{R}$ such that $|X| = |Y| = |Z| = n$ and*

$$|V \cap (X \times Y \times Z)| \geq cn^{2-\eta};$$

- (b) Let D denote the interval $(-1, 1)$. Then either V contains a cylinder over a curve $F(x, y) = 0$ or $F(x, z) = 0$ or $F(y, z) = 0$ or, otherwise, there are one-to-one analytic functions $f, g, h : D \rightarrow \mathbb{R}$ with analytic inverses such that V contains the $f \times g \times h$ -image of a part of the plane $x + y + z = 0$ near the origin:

$$V \supseteq \left\{ \left(f(x), g(y), h(z) \right) \in \mathbb{R}^3 ; x, y, z \in D ; x + y + z = 0 \right\};$$

- (c) The statement in (b) can be localised as follows. There is a finite subset $H \subset \mathbb{R}$ and an irreducible component $V_0 \subseteq V$ such that whenever $P \in V_0$ is a point whose coordinates are not in H , then one may require that $(f(0), g(0), h(0)) = P$.

This result indicates a significant “jump”: either V has the special form described in (b), in which case a quadratic order of magnitude is possible, by (b) \Rightarrow (c); or, else, we cannot even exceed $n^{2-\eta}$, by (a) \Rightarrow (b).

4. THEOREMS ON CURVES

Here we present some results on point sets located on algebraic curves and satisfying certain requirements.

The first one (Theorem 4.1) is a “gap version” of Theorem 2.2. It states that there is a significant difference between cubics and other algebraic curves: on a cubic, n points can determine as many as cn^2 triple lines; otherwise even as few as $n^{2-\eta}$ are impossible for n large enough.

The other result is related to a problem of Erdős. He asked if a point set with cn^2 quadruple lines must also contain a five-in-a-line. In Theorem 4.3 we settle this in the affirmative, under the additional assumption that the points lie on an algebraic curve.

Finally, Theorem 4.4 concerns point sets which determine few distinct directions.

Many triple lines force cubics. Our first main result states that, of all algebraic curves, only cubics can accommodate n points with $cn^{2-\eta}$ triple lines. This is probably far from being best possible; perhaps even the existence of as few as $cn^{1+\delta}$ such lines will also imply the same statement, for any $\delta > 0$ and $n > n_0(c, \delta)$.

Theorem 4.1. *For every $c > 0$ and positive integer d there exist $\eta = \eta(c, d)$ and $n_0 = n_0(c, d)$ with the following property. Let $\Gamma_1, \Gamma_2, \Gamma_3$ be (not necessarily distinct) irreducible algebraic curves of degree at most d in the plane \mathbb{R}^2 . Assume that $n > n_0$ and*

- (i) *no two Γ_i are identical straight lines;*
- (ii) *$\mathcal{H}_i \subset \Gamma_i$ with $|\mathcal{H}_i| \leq n$ ($i = 1, 2, 3$);*
- (iii) *$|\overline{\mathcal{H}_1 \mathcal{H}_2 \mathcal{H}_3}| \geq cn^{2-\eta}$.*

Then $\Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ is a cubic.

Remark 4.2. If we have an *arbitrary* (i.e., possibly reducible) algebraic curve Γ of degree d and a point set \mathcal{H} with many triple lines on it, then by the Pigeonhole Principle, some (at most three) irreducible components of Γ will contain a subset of \mathcal{H} which still determines at least $|\mathcal{H}|/d^3$ distinct triple lines. Therefore, the union of these components must be a cubic, according to the aforementioned Theorem.

Proof of Theorem 4.1. Let the curves $\Gamma_1, \Gamma_2, \Gamma_3$ be defined by the polynomial equations $F_1(x, y) = 0$, $F_2(x, y) = 0$, $F_3(x, y) = 0$, respectively. Three points $P_i(x_i, y_i) \in \Gamma_i$ ($i = 1, 2, 3$) are collinear iff

$$F(x_1, y_1, x_2, y_2, x_3, y_3) \stackrel{\text{def}}{=} \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix} = 0.$$

Eliminating the y_i from the system of the four equations

$$(2) \quad \{F(x_1, y_1, x_2, y_2, x_3, y_3) = 0\} \cup \{F_i(x_i, y_i) = 0 \mid (i = 1, 2, 3)\},$$

we get a polynomial relation $f(x_1, x_2, x_3) = 0$. In other words, the projection to \mathbb{R}^3 (i.e., to the subspace spanned by the x_i coordinates) of the two dimensional algebraic variety defined by (2) in \mathbb{R}^6 , will be contained in the zero-set of a single polynomial equation $f(x_1, x_2, x_3) = 0$.

Let $\eta = \eta(c, d)$ be as in Theorem 3.18. Denoting the set of the x coordinates of \mathcal{H}_i by X_i ($i = 1, 2, 3$), we have that the surface $S_f = \{f = 0\}$ contains at least $cn^{2-\eta}$ points of $X_1 \times X_2 \times X_3$.

In other words, (a) of the Surface Theorem 3.18 is satisfied for $V = S_f$ and the X_i . Since S_f cannot contain a cylinder by assumption (i), there exists an irreducible component $V_0 \subset S_f$ for which also (b) — localised as in (c) of the same Theorem — holds.

Pick a generic point $P(a_1, a_2, a_3) \in V_0 \subset S_f$. By the definition of the surface, there exist $b_1, b_2, b_3 \in \mathbb{R}$ such that, on the one hand, $Q_i(a_i, b_i) \in \Gamma_i$ for $i = 1, 2, 3$, while on the other hand, these Q_i are collinear. We can also assume without loss of generality, that these three points are distinct, they are regular points of $\Gamma_1 \cup \Gamma_2 \cup \Gamma_3$, and the straight line l which contains them is not tangent to Γ_i at Q_i ($i = 1, 2, 3$). [Indeed, V_0 is two dimensional by Theorem 3.18(b) while the points to be excluded form a finite number of one dimensional curves.]

Moreover, by (b) and (c) there, collinearity between sufficiently small arcs of the Γ_i around the Q_i is described by $(\mathbb{R}, +)$. Now if we rotate and/or shift the plane so that l becomes the y axis then, according to Remark 3.5, in a sufficiently small neighbourhood of 0, the (rotated) Γ_i coincide with the graphs of a standard system of continuous functions. Thus we can use Lemma 3.8 to conclude that a suitable cubic \mathcal{C} contains a non-empty open arc of each Γ_i . Thus also the union $\Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ of the three *irreducible* curves is contained in \mathcal{C} .

Finally, they cannot all be contained in a curve of degree < 3 since in that case they could not define many triple lines. Therefore, $\Gamma_1 \cup \Gamma_2 \cup \Gamma_3 = \mathcal{C}$. ■

Four-in-a-line. Erdős [5] posed the problem whether a set of n points which contains cn^2 collinear four-tuples must also contain five collinear points. To our best knowledge, no progress has been made on this question so far.

In 1995, M. Simonovits asked the following. *Is it possible to find n points on an irreducible algebraic curve of degree 4 which determine cn^2 four-in-a-line?* (Of course, such a set can contain no five-in-a-line.) We show here that the answer is in the negative, even in a more general setting.

Theorem 4.3. *If an algebraic curve Γ of degree d accommodates a set \mathcal{H} of n points with $cn^{2-\eta}$ distinct quadruple lines, where $\eta = \eta(c, d)$ is the same as in*

Theorem 4.1, then Γ contains four straight lines, each with $\geq c'(c, d) \cdot n^{1-\eta}$ points of \mathcal{H} , provided that $n > n_0(c, d)$.

Proof Γ has at most d irreducible components. Classify the $cn^{2-\eta}$ collinear four-tuples (located on distinct straight lines) according to which point lies on which component. By the Pigeonhole Principle, some four (not necessarily distinct) components $\Gamma_1, \Gamma_2, \Gamma_3$ and Γ_4 generate $cn^{2-\eta}/d^4 = c'(c, d)n^{2-\eta}$ quadruple lines. By Theorem 4.1, any three of the Γ_i must form a cubic. However, this is only possible if they are distinct straight lines. ■

Few directions. In [3], it was shown that if the graph of a polynomial $f \in \mathbb{R}[x]$ contains n points whose $\binom{n}{2}$ connecting lines only determine a linear number (at most Cn) distinct directions then the polynomial f is quadratic. (Some historic remarks and earlier results concerning sets which determine few directions can also be found there.)

Here we extend this to general algebraic curves.

Theorem 4.4. *For every $C > 0$ and positive integer d there is an $n_0 = n_0(C, d)$ with the following property.*

Let Γ_1 and Γ_2 be two (not necessarily distinct) irreducible algebraic curves, $n > n_0$, and $\mathcal{H}_i \subset \Gamma_i$ with $|\mathcal{H}_i| = n$ ($i = 1, 2$). Assume that among the directions of the straight lines $\overline{P_1P_2}$, for $P_i \in \mathcal{H}_i$ and $P_1 \neq P_2$, at most Cn are distinct. Then $\Gamma_1 \cup \Gamma_2$ is a (possibly degenerate) conic.

Proof Let Γ_3 be the line at infinity and \mathcal{H}_3 the set of the $\leq Cn$ directions on it. (If someone prefers no points at infinity, they can apply a projective mapping before proceeding further.) By assumption, $|\overline{\mathcal{H}_1\mathcal{H}_2\mathcal{H}_3}| \geq \binom{n}{2} > n^{2-\eta}$ if n is large. Hence, by Theorem 4.1, $\Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ is a cubic. Therefore, $\Gamma_1 \cup \Gamma_2$ is a conic. ■

5. STRAIGHT LINES AND CONICS

Theorem 5.1. *Let $n \leq |\mathcal{H}_1|, |\mathcal{H}_2|, |\mathcal{H}_3| \leq Cn$ and assume that \mathcal{H}_1 and \mathcal{H}_2 lie on the distinct straight lines l_1 and l_2 , respectively, while $\mathcal{H}_3 \cap l_1 = \mathcal{H}_3 \cap l_2 = \emptyset$.*

*If, moreover, $|\overline{\mathcal{H}_1\mathcal{H}_2\mathcal{H}_3}| \geq cn^2$, then some c^*n of the points of \mathcal{H}_3 , too, must be collinear. (Here $c^* = c^*(c, C)$ does not depend on n .)*

Proof Apply a projective transform π which maps l_1 to the line at infinity. Then some cn^2 pairs of points of $\pi(\mathcal{H}_2) \times \pi(\mathcal{H}_3)$ determine at most $|\pi(\mathcal{H}_1)| = |\mathcal{H}_1| \leq Cn$ distinct directions, while $\pi(\mathcal{H}_2)$ is still collinear. By a result in [1] (see Theorem 3 there), also $\pi(\mathcal{H}_3)$ — hence \mathcal{H}_3 , too — must contain c^*n collinear points. ■

The following Theorem 5.2 is the “elder brother” of Theorem 5.1 in the sense that now we start from a non-degenerate conic while the two lines l_1, l_2 above can be considered as a degenerate one.

Theorem 5.2. *Let $C > 1$ be arbitrary and $\mathcal{H}_1, \mathcal{H}_2 \subset \mathbb{R}^2$. Assume that*

- (a) $n \leq |\mathcal{H}_1|, |\mathcal{H}_2| \leq Cn$;
- (b) \mathcal{H}_2 lies on a non-degenerate conic which contains no point of \mathcal{H}_1 ;
- (c) $|\overline{\mathcal{H}_1\mathcal{H}_2}| \geq n^2$.

*Then some c^*n of the points of \mathcal{H}_1 must be collinear (where $c^* = c^*(C)$ does not depend on n .)*

Proof First, without loss of generality, we may assume that every point of \mathcal{H}_1 is incident upon at least n triple lines. (Otherwise keep on deleting those with less than $n/(2C)$ such lines and finally, use the new values of $n' = n/(2C)$, $C' = 2C^2$.)

Moreover, we may assume that the conic which contains \mathcal{H}_2 , is the parabola $y = x^2$. (Else we apply a projective mapping which maps it to that curve. This can also be done such a way that no point of \mathcal{H}_1 is mapped to the line at infinity and the x -coordinates of the points in $\mathcal{H}_1 \cup \mathcal{H}_2$ become all distinct.)

Denote the coordinates of the points of \mathcal{H}_1 by (a_i, b_i) and the set of the x -coordinates of the points of \mathcal{H}_2 by X , i.e.,

$$\mathcal{H}_1 = \{(a_i, b_i) \mid i = 1, 2, \dots, |\mathcal{H}_1|\};$$

$$\mathcal{H}_2 = \{(x, x^2) \mid x \in X\},$$

where, of course, $|X| = |\mathcal{H}_2|$.

Proposition 5.3. *Two distinct points (x, x^2) , (y, y^2) of \mathcal{H}_2 and a point $(a_i, b_i) \in \mathcal{H}_1$ are collinear iff*

$$xy - a_i x - a_i y + b_i = 0. \blacksquare$$

The above equations can be considered as functions of type $X \mapsto X$:

$$y = f_i(x) \stackrel{\text{def}}{=} \frac{a_i x - b_i}{x - a_i}.$$

These projective mappings f_i are “vertical projections” (to X) of the involutions of the parabola, with centres (a_i, b_i) .

We started with the assumption that every point of \mathcal{H}_1 is incident upon at least n triple lines. Therefore, each f_i maps at least n elements of X to elements of X . According to [2] Theorem 29 (the “Image Set Theorem”), some c^*n of the f_i must be collinear — if we represent them as elements of the three dimensional projective space. In other words, in that space at least c^*n points of projective coordinates $(a_i, -b_i, 1, -a_i)$ are combinations of as few as two of them, say $(a_1, -b_1, 1, -a_1)$ and $(a_2, -b_2, 1, -a_2)$. Considering the (constant) third coordinates, this is only possible if — even as four dimensional vectors — $(a_i, -b_i, 1, -a_i) = \lambda_i(a_1, -b_1, 1, -a_1) + (1 - \lambda_i)(a_2, -b_2, 1, -a_2)$, for suitable reals λ_i . We conclude that also the corresponding c^*n original points $P_i(a_i, b_i) \in \mathcal{H}_1 \subset \mathbb{R}^2$ must be collinear. \blacksquare

6. CONCLUDING REMARKS

Beyond Conjecture 2.1 the following remain open.

Problem 6.1. Let $\delta > 0$ be arbitrary. Does the conclusion “ $\Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ is a cubic” of Theorem 4.1 hold if, in place of (iii), we only assume

$$(iii^*) \quad |\overline{\mathcal{H}_1 \mathcal{H}_2 \mathcal{H}_3}| \geq n^{1+\delta}$$

— provided that $n > n_0 = n_0(\delta, d)$?

Problem 6.2. Does Theorem 4.3 hold with $n^{1-\eta/2}$ in the statement (in place of $n^{1-\eta}$)?

Problem 6.3. Let $\delta > 0$ be arbitrary. Does the conclusion “ $\Gamma_1 \cup \Gamma_2$ is a conic” of Theorem 4.4 hold if we only assume that the lines $\overline{P_1 P_2}$ only determine $\leq n^{2-\delta}$ distinct directions — in place of Cn — provided that $n > n_0 = n_0(\delta, d)$?

Acknowledgements

We are grateful to Endre Makai for his very constructive comments on (and simplifications to) some earlier versions of the manuscript and also to Zoltán Járαι for sharing with us his typesetting \TeX expertise.

REFERENCES

- [1] Gy. Elekes. On linear combinatorics III. *Combinatorica*, 19(1):43–53, 1999.
- [2] Gy. Elekes and Z. Király. On the combinatorics of projective mappings. *Journal of Algebraic Combinatorics*, 14(3):183–197, 2001.
- [3] Gy. Elekes and L. Rónyai. A combinatorial problem on polynomials and rational functions. *Journal of Combinatorial Theory, series A*, 89:1–20, 2000.
- [4] Gy. Elekes and E. Szabó. How to find groups? (and how to use them in Erdős geometry?). *Combinatorica*, 32(5):537–571, 2012.
- [5] P. Erdős and G. Purdy. Some extremal problems in geometry IV. In *Proc. 7th Southeastern Conference Combinatorics, Graph Th. and Comp.*, pages 307–322, 1976.
- [6] W. Fulton. *Algebraic Curves*. W.A. Benjamin Inc., New York – Amsterdam, 1969.
- [7] B. Green and T. Tao. On sets defining few ordinary lines. preprint: arXiv:1208.4714.
- [8] J. Jackson. *Rational Amusements for Winter Evenings*. Longman Hurst Rees Orme and Brown, London, 1821.
- [9] R. E. Jamison. Planar configurations which determine few slopes. *Geometriae Dedicata*, 16:17–34, 1984.
- [10] M. Reid. *Undergraduate Algebraic Geometry*. Cambridge University Press, 1988.
- [11] J. J. Sylvester. Problem 2473. *Math. Questions from the Educational Times*, 8:106–107, 1867.
- [12] F. Wetttl. On the nuclei of a point set of a finite projective plane. *Journal of Geometry*, 30(2):157–163, 1987.

GYÖRGY ELEKES, EÖTVÖS UNIVERSITY, BUDAPEST
E-mail address: `elekes@cs.elte.hu`

ENDRE SZABÓ, RÉNYI INSTITUTE OF THE HUNGARIAN ACADEMY OF SCIENCES
E-mail address: `endre@renyi.hu`